Modeling Interaction of Insurgency and Counterinsurgency by Differential Equations

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Abstract

In this paper, we first summarize the Lanchester laws and related applications to combat warfare and phase II counterinsurgency. Then, we examine the possibilities of modeling the interactions between counterinsurgency and insurgency by applying ecology models known as Lotka-Volterra systems. Two kinds of models, known as predator-prey and interference COIN competition systems, are proposed based on the principles and guidelines deliberated by Drapeau et al (2008). Preliminary investigations on the criteria of coexistence and extinction are conducted. Contrary to the Competition Exclusion Principle (CEP), counterinsurgency and insurgency forces could coexist for a relatively long period, provided that the population capacity for the rebellion force is high (equivalently, the authority response is low), or the competition level is weak. Computer simulations by Mathematica are conducted for several examples. In the last section, general models with possible response functions for future case studies are proposed.

Brief introduction to modeling combat warfare by differential equations

Modeling combat battles by differential equations can be dated back to1916 when the British engineer F. W. Lanchester modeled the aerial combat in WWI by applying differential equation systems which had been created to describe biological phenomena. Since then the Lanchester models have been developed into different deterministic forms and applied to the area of warfare modeling.

It is noted that Lanchester deterministic models are applied to the current study of mobility capability and requirements (MCRS). Parameters and coefficients in MCRS Lanchester models are determined by stochastic process. Computer simulations of related Lanchester models by Mobility Simulator are conducted to provide insight view of warfare scenarios.

In this section, we will briefly introduce the terms and theory of Lanchester deterministic systems and certain research findings in this area.

Of the two combat sides, the side whose population reaches zero first would be considered the loser. There are two basic laws known as the Lanchester Square Law (aimed fire encounter) and Lanchester Linear Law (un-aimed fire situation).

Lanchester Square Law (Aimed Fire)

Consider the system of differential equations

$$\begin{aligned} x'(t) &= -ay \\ y'(t) &= -bx \end{aligned} \qquad (1)$$

where a and b are force y's and force x's lethality coefficients, respectively. Assume that both populations x and y are positive at the beginning, i.e., the initial values $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$. By separating the variables, we obtain an implicit solution equation $a(x^2 - x_0^2) = b(y^2 - y_0^2)$ or $ax^2 - by^2 = ax_0^2 - by_0^2$

If $ax_0^2 \le by_0^2$, then $ax^2 - by^2 \le 0$ all time and $\lim_{t\to\infty} x(t) = 0; \lim_{t\to\infty} y(t) = \sqrt{y_0^2 - \frac{a}{b}x_0^2}$. Force y wins the war.

If $ax_0^2 \ge by_0^2$, then $ax^2 - by^2 \ge 0$ all time and $\lim_{t\to\infty} y(t) = 0; \lim_{t\to\infty} x(t) = \sqrt{x_0^2 - \frac{b}{a}y_0^2}$. Force x wins the war.

This simulation concludes that the winner of a two-sided war (aimed fire) is determined by comparing the initial combat populations along with the lethality of both sides.

Lanchester's Linear Law (un-aimed fire)

If x's unit can fire at rate r, then attrition suffered by y is
$y'(t) = -(ra/A)xy$
where a is area within area A where shooting occurs and $\alpha = ra/A$ is the lethality coefficient for x force. Similarly we can set up an equation for the rate of change in x force and eventually a system of equations as follows

$$\begin{aligned} x'(t) &= -\beta xy \\ y'(t) &= -\alpha xy \end{aligned} \qquad (2)$$

where β is the lethality of y force. The solution for the system is
$\alpha x - \beta y = \alpha x_0 - \beta y_0$
A reasonable assumption is that both populations start positive, leading to the conclusion that $\lim_{t\to\infty} y(t) = 0; \lim_{t\to\infty} x(t) = (x_0 - \beta/\alpha)y_0$, provided that the product of the lethality and the initial x force is greater than that of y, i.e., $\alpha x_0 > \beta y_0$

Engel (1954) applied the following Lanchester system to the battle of Iwo Jima and estimated the parameters using actual data
$x'(t) = r(t) - \beta y; x > 0$
$y'(t) = -\alpha x$
where x and y are US and Japanese troops in the island and $r(t)$ is the rate at which US troops were landed. The parameters α and β have the units of "opposing casualties per man day of combat," and were chosen to fit the record of what actually happened. If x is defined to be US troops not killed, wounded, or missing," the best fit is with $\alpha = 0.0544$ US casualties per Japanese man day and $\beta = 0.0106$ Japanese casualties per US man day. The model also fits the case in which x is defined as "US troops alive".

Engel also did similar estimates for the battle of Crete, concluding that there were 0.104 Allied casualties per German man day and 0.0162 German casualties per Allied man day. The Crete parameters are not grossly different from the Iwo Jima parameters that held at a different place at a different time in a different part of the world. It is noted that earlier than Engel, the Russian M. Osipov did considerable work on estimating the parameters for land battles.

A realistic model of the WWII U-boat war in the Atlantic assumes that submarines operate only in “wolf packs” and that all ships sail in escorted convoys. Morse and Kimball (1950) make the approximation that $5n/c$ merchant ships and $nc/100$ U-boats, will be lost when a convoy defended by c escorts is attacked by n submarines. Since the exchange ratio of U-boats to escorts was about 5/1, we will also assume that the average number of escorts lost is $nc/500$. Greg Brown (2001) considered the destruction efficiency and adopted the following system of equations:

$$\frac{dS}{dt} = -e\left(\frac{nc}{100}\right) - f_s + R_s \text{ , actual rate of submarine destruction}$$

$$\frac{dE}{dt} = -e\left(\frac{nc}{500}\right) + R_e \text{, the actual rate of caravan escort destruction}$$

$$\frac{dM}{dt} = e\left(\frac{5n}{c}\right) \text{, the rate of the merchant ships being destroyed}$$

Using different sets of initial conditions $M_0 = 0, S_0 = 50, E_0 = 100$ and $M_0 = 0, S_0 = 500, E_0 = 100$, he simulated the cases by running Matlab and concluded that over time, regardless of the initial amount of submarines, the allies will prevail in the Atlantic theater.

Generalized Lanchester Theory for Phase II insurgency

The research results mentioned above are all about combat wars within short time periods. Schaffer (1965) generalized the Lanchester models and applied them to model Phase II Insurgency, developing corresponding deterministic forms of Lanchester’s equations.

According to Mao (1954), the first two phases of insurgency are characterized by ground-yielding operations on the part of the insurgents; in the final phase, the insurgents take the strategic offensive. Phase II is generally a period of strategic stalemate. The insurgent operations become increasingly military in nature but the conflict remains localized. As necessary, the insurgents continue to yield ground for time and eventual strength.

Taking into consideration that there are three types of force depletion (casualties): KIA, WIA, surrender, and desertion, we assume that the total force depletion rate is the sum of the rates from each of these sources.

Assume that both sides are permitted supporting weapons. No supporting weapons duels are allowed. It is presumed that the insurgents disengage rather than participate in this type of activity.

$$\frac{dm_c}{dt} = -k_n(t)n - \sum_i K_i(t)w_i$$
$$\frac{dn_c}{dt} = -k_m(t)m - \sum_j K_j(t)w_j \qquad (3)$$

where *m* and *n* are the number of engaged personnel on opposite sides, W_i and W_j are the supporting weapon strengths, and t is a time-like variable. It is noted that, in general, the weapon efficiency coefficients, k_n, k_m, K_i and K_j are positive functions of time.

M. B. Schaffer studied three cases of Phase II insurgency and established corresponding systems of differential equations for each case.

In the case of a skirmish (with no surprise), with certain assumptions for computer simulations for N-force (time in minutes), M. B. Schaffer approached the solutions of the corresponding system for skirmish. The conclusions are charted to illustrate casualties. For example,

(1) With different initial values $N - force = 50; M - force = 100$, N-force is dropped to 15 in 5.5 minutes.
(2) With the same initial value $N - force = M - force = 50$, N-force dropped to 20 in 15 minutes.

In contrast to the skirmish, the ambush (with surprise factor) represents a case where the time dependence of the weapon efficiency coefficients is important, and perhaps dominant. This time dependency results from the changing cover (shielding) are available to the individuals of the defensive side. In an ambush, due to the surprise element, defensive cover is initially at its minimum. As the engagement progresses, the defense seeks whatever cover is available and gradually improves this situation. The attackers have a relatively secure position which remains constant until they choose to break off the engagement, since there is little motivation for defections or surrenders on the attacking side, and since the defense cannot bring its supportive weapons into play in the early stages of the fight.

With certain assumptions for computer simulations, Schaffer successfully simulated ambush using computer software. For examples,

(1) initial attacker force = 50, initial defender force = 100 (defender remaining force dropped from 100 to 87 in 42 min; attacker remaining force from 50 to 18)
(2) initial attacker force = 50, initial defender force = 100 (defender remaining force dropped from 50 to 10 in 36 min; attacker remaining force from 50 to 26)

In the third case, Schaffer divided siege into to three stages: (1) an initial "softening-up" phase where support weapons are primary (the riflemen are generally out of range), and (2) an assault stage where the offensive artillery barrage must be lifted. Actual solutions are given in this case.

Rationale on adopting ecological models to simulate COIN

The success of Lanchester models on warfare is an inspiration for the task of modeling COIN by differential equations. We can inherit the definition of the logistic equation and terms such as lethality. However, we should be aware that the Lanchester theories concentrate on working for short-term warfare. Schaffer's generalizations made one step forward toward modeling Phase II insurgency but focusing on Phase II short-term combats. Lanchester-type case studies show that one side of a combat declines to zero eventually. Therefore, solutions of Lanchester systems are decreasing all time to the end of a war.

The interactions of insurgency and counterinsurgency are more complicated than that in short-term warfare. The Lanchester laws may not be appropriate to describe the rise and fall of insurgency for a relatively long period of time.

Drapeau el al (2008) deliberated the possibilities of simulating the interaction of insurgency and counterinsurgency by applying ecology models known as Lotka-Volterra predator-prey systems.

In a COIN ecosystem, we can treat authority as predator and rebellion as prey within an isolated country or district setting. Questions are raised on whether or not such predator-prey models could accurately depict the interactions and relationships between authority and rebellion in a COIN ecosystem. Although such approaches may be too simple to simulate a real situation, it is worth to establish models and to analyze COIN situations using mathematical outcomes such as persistence. In addition, the traditional Lotka-Volterra model has been since 1920's modified in a number of forms, including continuous or discrete, constant or periodic, instantaneous or delayed and etc. Predator-prey models have been generalized to other areas, for instance, biomedicine, economics and industry. Models from multiple resources could be applied to simulate COIN.

There are three population variables in consideration: Authority (A), Rebellion (R), Population (P) with the following ecological scenario: **A preys on R with counter attack R on A; both A and R compete for access to P** (a precursor to winning support: a mean to an end). Coexistence of the competitors, thought unwelcome, could be discussed and used for strategic planning purpose.

Logistic Equations and Environmental Capacity

We will introduce and define related basic terms for modeling COIN in this section.

The basic concept in a population is the intrinsic rate of growth – the per capita growth rate. Let $x(t)$ be the number of members of a population at time *t*. Then the intrinsic rate of growth of $x(t)$ is $x'(t)/x(t)$.

The simplest possible model (Malthus) would be assuming that the intrinsic growth rate be a constant, that is,

$$\frac{x'(t)}{x(t)} = r, r \geq 0 \text{ or equivalently } x'(t) - rx(t) = 0 \qquad (4)$$

The solutions for this differential equation have the general form $x(t) = x(t_0)e^{r(t-t_0)}$. If $r \neq 0$, this model indicates unbounded exponential growth. One interpretation about the model is that *r* is the difference of the birth rate and death rate of the population so that the two quantities are balanced to avoid unlimited growth. However this model does not serve on long-term prediction of a population.

Taking into account the environmental capacity for a population, a more reasonable model is

$$x'(t) = rx\left(1 - \frac{x}{K}\right) \tag{5}$$

where *K* is called the carrying capacity of the environment and *r* the maximal growth rate. An explicit general solution can be obtained by separating variables. It is easy to observe from the equation that the growth rate $x'(t)$ is slowed down when the population $x(t)$ is approaching the environmental carrying capacity. We can solve the equation by separating the variables and obtain $x(t) = \dfrac{x_0 K e^{r(t-t_0)}}{K + x_0[e^{r(t-t_0)} - 1]}$, where $x_0 = x(t_0)$. In fact, $\lim_{t\to\infty} x(t) = K$ eventually.

Considering the rebellion insurgency in a population, the simplest logistic growth for the rebellion force is then modeled by $R'(t) = r(1 - R/K)$, where *r* is the difference of recruiting rate and depletion rate (including KIA, WIA, surrender, and desertion), *R* represents the density of insurgents and *K* is the maximum density of recruits the population can provide.

We understand that $R(t) \leq K$ and therefore $R'(t) \geq 0$ all time. With no presence of predator force, the density of insurgency is gradually growing to its maximum, but no more than its maximum since the growth rate $R'(t) = 0$ stops growing as $R = K$. This math discovery could be supported by cases in which rebellion got absolute control. For example, with the retreat of US forces in Vietnam, Viet Cong took over the power and mind controlled the entire country population in a short time period. Another example would be Taliban's sweep when the Soviet Union gave up its occupation of Afghanistan.

Lotka-Volterra Models for COIN

We must point out that the growth of a rebellion force is based on recruiting efforts. Therefore, revisions are necessary as we model a COIN situation.

With the presence of the predator (authority) force, the growth rate of the rebellion force is under control or slowed down. As a simple approach, the interactions between the two sides can be modeled by the following system of equations

$$\begin{aligned} \frac{dR}{dt} &= R(a - bR - cA) \\ \frac{dA}{dt} &= A(-d + eR) \end{aligned} \tag{6}$$

where all constants a, b, c, d and e are assumed positive. Specifically, a is the intrinsic growth rate of $R(t)$ without the presence of $A(t)$ and $K = b/a$, the population maximal supportive capacity, c is the controlling (lethality) rate by $A(t)$, d is the decreasing rate of $A(t)$ without $R(t)$, and e is the responding rate of $A(t)$ to the presence of $R(t)$.

In this model, if there is no threat to the authority, the equation for the growth rate $R'(t)$ is the same as the logistic equation but affected as the authority is present. From the second equation, the authority force is extinct if there is no rebellion force. It means the authority force demand is gradually decreased to zero, as the rebellion force disappears. On the other hand, the need for authority force is growing proportionally to the product of the densities of the authority and rebellion forces, featured by the response constant *e*.

Though counter-attack from rebellion force is not considered in this predator-prey type model, it is still meaningful taking into consideration the situation in which the authority has enough backup support to immediately fill up vacancies caused by rebellion's counter-attacks.

In case the response from the authority is delayed (information or operation time delay), we should include delay effects in our discussion by adopting delay-differential equation models. One simple revision from (6) is

$$\begin{aligned} R'(t) &= R(t)[a - bR(t) - cA(t)] \\ A'(t) &= A(t)[-d + eR(t-\tau)] \end{aligned} \qquad (7)$$

where constant $\tau \geq 0$ addresses the time delay in authority response caused by distance, communication, and/or operation.

Behavior around the equilibriums

A set of constant solutions is called an equilibrium point for an ordinary differential equation system. To find all equilibrium points, we set the right sides of System (6) to be 0, i.e.,

$$\begin{aligned} R(a - bR - cA) &= 0 \\ A(-d + eR) &= 0 \end{aligned} \qquad (8)$$

Solving the linear equations, we obtain three equilibrium points $(0,0)$, $(a/b, 0)$ and $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$.

The trivial equilibrium $(0,0)$ is a saddle point. The conclusion can be explained by the linear part of System (6)

$$\begin{aligned} \frac{dR}{dt} &= aR \\ \frac{dA}{dt} &= -dA \end{aligned} \qquad (9)$$

and its eigenvalues $\lambda_1 = a, \lambda_2 = -d$

For the equilibrium $(a/b, 0)$, we make a transformation $x = R - a/b, y = A$ for System (6) and obtain

$$\begin{aligned} \frac{dx}{dt} &= \left(x + \frac{a}{b}\right)\left[a - b\left(x + \frac{a}{b}\right) - cy\right] = -ax - \frac{ac}{b}y - bx^2 - cxy \\ \frac{dy}{dt} &= y\left[-d + e\left(x + \frac{a}{b}\right)\right] = \left(-d + \frac{ea}{b}\right)y + exy \end{aligned} \tag{10}$$

The behavior of the equilibrium $(0,0)$ (corresponding to $(a/b, 0)$) is determined by the eigenvalues $\lambda_1 = -a, \lambda_2 = -d + \frac{ea}{b} = e\left(\frac{a}{b} - \frac{d}{e}\right) = e\left(K - \frac{d}{e}\right)$ of its linear part

$$\begin{aligned} \frac{dx}{dt} &= -ax - \frac{ac}{b}y \\ \frac{dy}{dt} &= \left(-d + \frac{ea}{b}\right)y \end{aligned} \tag{11}$$

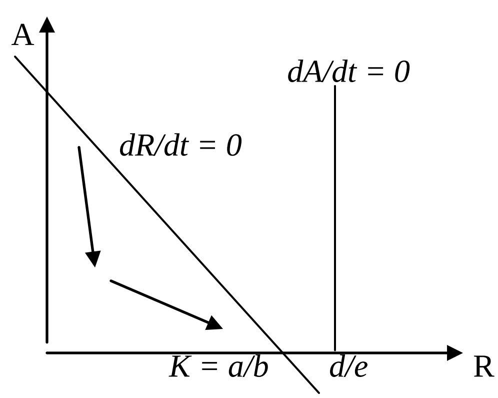
Case I. $\frac{a}{b} = K < \frac{d}{e}$. Both eigenvalues are negative. The equilibrium is a stable node (attractor).

Case II. $\frac{a}{b} = K = \frac{d}{e}$. As both eigenvalues are nonnegative, the equilibrium is again a stable node.

Case III. $\frac{a}{b} = K > \frac{d}{e}$. As the eigenvalues are opposite in signs, $(a/b, 0)$ is a saddle point.

We will illustrate the behavior $(0,0)$ and $(a/b, 0)$ as we study the possible third equilibrium point $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$. Since both population densities are nonnegative, there are three cases of the other equilibrium $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$.

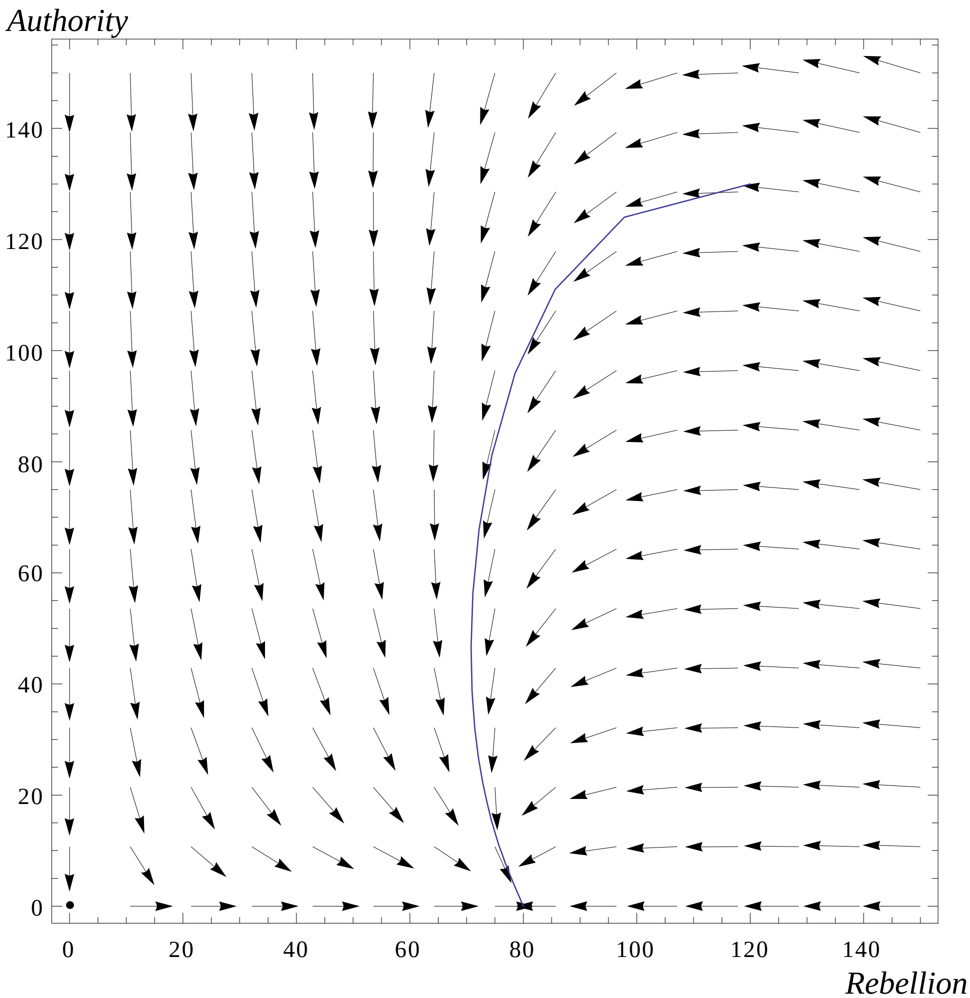
Case I. $\frac{a}{b} = K < \frac{d}{e}$. Then the system has only two nonnegative equilibrium points $(0,0)$ (saddle point) and $(a/b, 0)$ (stable node). In other words, if the authority responding ratio e is too small, the rebellion will defeat the authority and approaches its maximum.



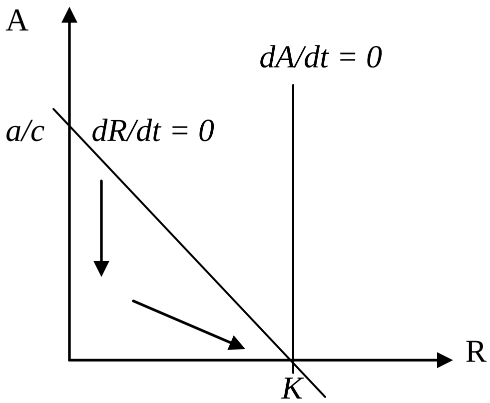
Example 1

$$\frac{dR}{dt} = R(4 - 0.05R - 0.01A)$$
$$\frac{dA}{dt} = A(-6 + 0.05R) \qquad (12)$$

$K = \frac{a}{b} = \frac{4}{0.05} = 80 < \frac{d}{e} = \frac{6}{0.05} = 120$. No positive equilibrium point. (0,0) and $(a/b,0) = (80,0)$ are the only nonnegative equilibrium points. The computer simulation is made possible by Mathematica. It shows all trajectories starting with positive values will eventually approaches $(80,0)$ as a stable node. In other words, if the authority response ratio (represented by e) is low, the rebellion force eventually approaches its maximal population acceptance.



Case II. $\frac{a}{b} = K = \frac{d}{e}$. The system has two nonnegative equilibriums, $(0,0)$ and $\left(\frac{d}{e}, 0\right)$.



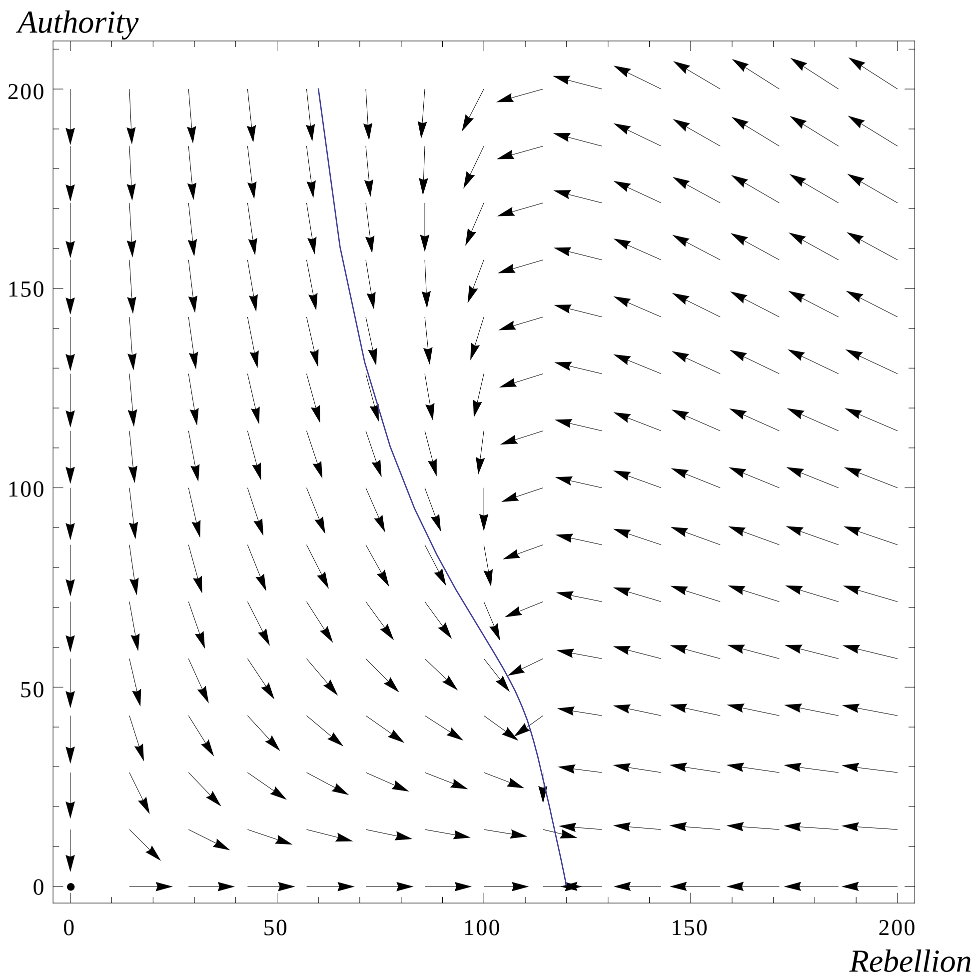
In this case, $dR/dt < 0$ all time and $dA/dt \begin{cases} < 0 \; if \;\; R < K \\ > 0 \; if \;\; R > K \end{cases}$ for any positive initial values (R_0, A_0). We conclude that both $R(t) \to K$ and $A(t) \to 0$ eventually, given sufficient amount of time.

Example 2

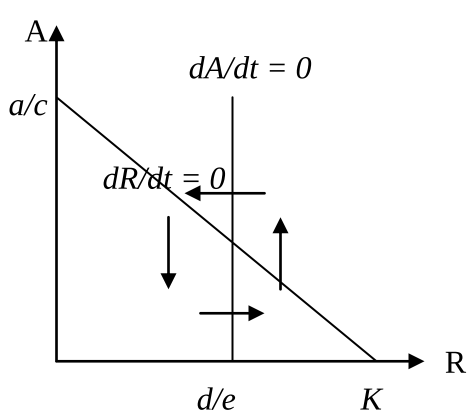
$$\begin{aligned} \frac{dR}{dt} &= R(6 - 0.05R - 0.01A) \\ \frac{dA}{dt} &= A(-6 + 0.05R) \end{aligned} \qquad (13)$$

$$\frac{a}{b} = K = \frac{a}{b} = \frac{6}{0.05} = \frac{d}{e} \text{ and} \left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right) = \left(\frac{6}{0.05}, \frac{0.05}{0.01}\left(\frac{6}{0.05} - \frac{6}{0.05}\right)\right) = (120,0)$$

There are only two equilibriums in this case: $(0,0)$ as a saddle point and $(120,0)$ as a stable node. The following simulation is made possible by Mathematica. The solid trajectory starts at $(R_0, A_0) = (60,200)$ and eventually approaches $(120,0)$. The vector field of the system shows that no matter what initial forces the two sides begin, the rebellion force approaches its maximal population acceptance.



Case III. $\frac{a}{b} = K > \frac{d}{e}$. There are three nonnegative equilibrium points $(0,0)$, $(a/b,0)$ and $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$. The population supportive capacity is greater than the authority's attacking effort.



In the plane, above the line $a - bR - cA = 0$, $dR/dt < 0$, and $dR/dt > 0$ below the line. On the right side of the vertical line $-d + eR = 0$, $dA/dt > 0$, and $dA/dt < 0$ on the left side of the vertical line. The behavior of the trajectories around the positive equilibrium $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$ is illustrated above. However, it is not clear whether the equilibrium is a center or an attractor.

We need to analyze the behavior of the trajectories near the positive equilibrium. To this end, we make transformation $x = R - \frac{d}{e}, y = A - \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)$ in System (6), and obtain a nonlinear system of *x* and *y*, after simplification

$$\begin{aligned} \frac{dx}{dt} &= -\frac{bd}{e}x - \frac{cd}{e}y - bx^2 - cxy \\ \frac{dy}{dt} &= \left(\frac{ae}{c} - \frac{bd}{c}\right)x + exy \end{aligned} \tag{14}$$

which has (0, 0) as its equilibrium point corresponding to the positive equilibrium $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$ of System (6). As a nonlinear system, the behavior of (0,0) is controlled by its linear part

$$\begin{aligned} \frac{dx}{dt} &= -\frac{bd}{e}x - \frac{cd}{e}y \\ \frac{dy}{dt} &= \left(\frac{ae}{c} - \frac{bd}{c}\right)x \end{aligned} \tag{15}$$

The characteristic equation of System (15)

$$\det\begin{bmatrix} -\frac{bd}{e} - \lambda & -\frac{cd}{e} \\ \left(\frac{ae}{c} - \frac{bd}{c}\right) & -\lambda \end{bmatrix} = \lambda^2 + \frac{bd}{e}\lambda + ad - \frac{bd^2}{e} = 0 \tag{16}$$

has the following two roots

$$\lambda = \frac{\frac{-bd}{e} \pm \sqrt{\left(\frac{bd}{e}\right)^2 - 4\left(ad - \frac{bd^2}{e}\right)}}{2}$$

Since $\frac{a}{b} = K > \frac{d}{e}$, we conclude that both eigenvalues of the characteristic equation have negative real parts and consequently the equilibrium point is an attractor. In other words, it is asymptotically stable.

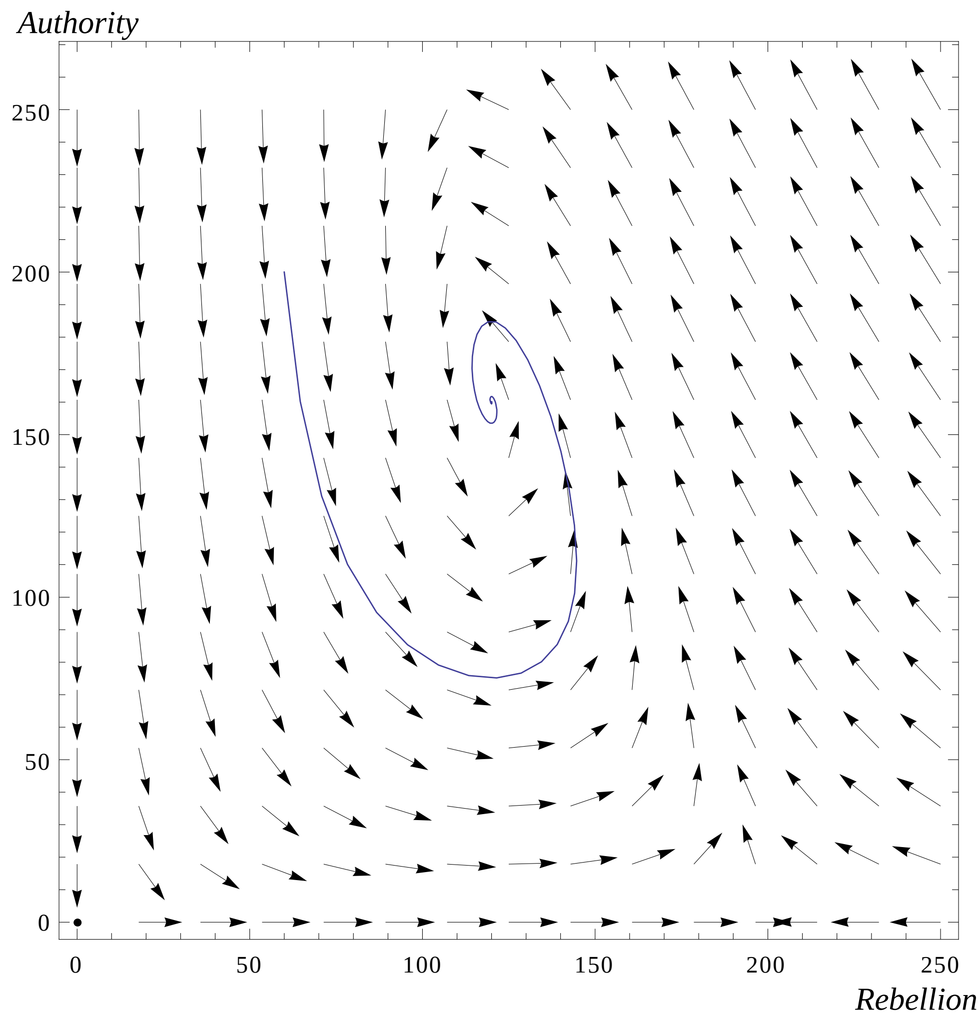
Therefore, the rebellion and the authority forces will stay coexist approaching to its equilibrium in long-term run.

Example 3

$$\frac{dR}{dt} = R(4 - 0.02R - 0.01A)$$
$$\frac{dA}{dt} = A(-6 + 0.05R) \qquad (17)$$

The positive equilibrium is $\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right) = \left(\frac{6}{0.05}, \frac{0.02}{0.01}\left(\frac{4}{0.02} - \frac{6}{0.05}\right)\right) = (120,160)$ is a stable node. This is a case with a high population supportive capacity.

The following computer simulation for Example 3 is made possible by Mathematica. The trajectory is the solution curve with initial rebellion force = 60 and authority force = 200. The vector field shows that all possible trajectories if started with non-zero force will eventually approach the equilibrium (120,160). The other two equilibrium points $(0,0)$ and $(a/b,0) = (200,0)$ are both saddle points.



We notice in Case III that the existence of rebellion is independent of the controlling (lethality) efforts by the authority, represented by *c*. It could be interpreted that the phenomenon of the rebellion existence lasts for a relatively long time period if the population's supportive acceptance, represented by *K*, of the rebellion is high enough.

We summarize our findings in the following table.

Table 1. Behavior near equilibriums

	$\frac{a}{b} = K < \frac{d}{e}$	$\frac{a}{b} = K = \frac{d}{e}$	$\frac{a}{b} = K > \frac{d}{e}$
$(0,0)$	**Saddle point**	**Saddle point**	**Unstable node**
$(a/b,0)$	**Stable node**	**Stable node**	**Saddle point**
$\left(\frac{d}{e}, \frac{b}{c}\left(\frac{a}{b} - \frac{d}{e}\right)\right)$	**Non-positive**	**Same as** $(a/b,0)$	**Stable node**

Interference Competition Models

Another kind of ecological models favored by Drapeau et al (2008) is called interference competition models. They conjecture that this model may be more useful than Lotka-Volterra models for describing the complex conflict ecosystem of COIN.

Assume both authority (A) and rebellion (R) compete for access to the neutral population (P). Similar to the predator-prey setups, we can establish the following competition system to simulate the interaction between the variables A and R.

$$\frac{dR}{dt} = R(b_1 - a_{11}R - a_{12}A)$$
$$\frac{dA}{dt} = A(b_2 - a_{21}R - a_{22}A) \qquad (18)$$

where all constant coefficients are assumed positive. Define $K_i = b_i / a_{ii}, i = 1,2$, then each can be seen as the population's maximal acceptance for the competing component without the presence of its opponent. a_{12} can be considered as the rates of lethality of A on R and a_{21} is the lethality of R on A, respectively. These two rates reflect the level of competition. Small a_{12} and a_{21} reflect low competition conflict.

Simple comparisons $\frac{dR}{dt} \le R(b_1 - a_{11}R)$ and $\frac{dA}{dt} \le A(b_2 - a_{22}A)$ lead to the conclusion that the population for each competitor is bounded.

There are four possible equilibrium points for System (18). Three are easily observed as $(0,0), (b_1 / a_{11}, 0), (0, b_2 / a_{22})$

Another possible equilibrium point is determined by whether or not the coefficient matrix $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ of the linear system $\begin{matrix} b_1 - a_{11}R - a_{12}A = 0 \\ b_2 - a_{21}R - a_{22}A = 0 \end{matrix}$ is invertible. If it's invertible, then the solution $\begin{bmatrix} R^* \\ A^* \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ is an equilibrium point, the only possible positive equilibrium. To avoid the trivial case, we assume that the matrix is invertible, in other words, the lines $b_1 - a_{11}R - a_{12}A = 0$ and $b_2 - a_{21}R - a_{22}A = 0$ are not parallel.

In the following, we will analyze the local behavior near each equilibrium, using the linearization of System (18).

(0,0) is a unstable node (repeller) as its linear system has two positive eigenvalues. It means as long as the competition goes on, a winner or coexistence will be determined. The case that both competitors are extinct at the same time will not happen.

At $(b_1/a_{11},0)$, we make a transformation in System (20) $x = R - b_1/a_{11}, y = A$ and obtain

$$\begin{aligned} \frac{dx}{dt} &= (x + b_1/a_{11})[b_1 - a_{11}(x + b_1/a_{11}) - a_{12}y] = -b_1 x - \frac{a_{12}b_1}{a_{11}} y - a_{11}x^2 - a_{12}xy \\ \frac{dy}{dt} &= y[b_2 - a_{21}(x + b_1/a_{11}) - a_{22}y] = (b_2 - a_{21}b_1/a_{11})y - a_{21}xy - a_{22}y^2 \end{aligned} \qquad (19)$$

The linear part for (0,0) corresponding to $(b_1/a_{11},0)$

$$\begin{aligned} \frac{dx}{dt} &= -b_1 x - \frac{a_{12}b_1}{a_{11}} y \\ \frac{dy}{dt} &= (b_2 - a_{21}b_1/a_{11})y \end{aligned} \qquad (20)$$

One of the eigenvalues $\lambda_1 = -b_1 < 0$ and the sign for the other one $\lambda_2 = b_2 - a_{21}b_1/a_{11}$ is pending in three cases

$$\lambda_2 = b_2 - a_{21}b_1/a_{11} \begin{cases} < 0, if\ b_2/a_{21} < b_1/a_{11}, equivalently, a_{21}/b_2 > a_{11}/b_1 \\ = 0, if \qquad b_2/a_{21} = b_1/a_{11} \\ > 0, if\ b_2/a_{21} > b_1/a_{11}, equivalenly, a_{21}/b_2 < a_{11}/b_1 \end{cases}$$

In the first and second cases, $\lambda_2 = b_2 - a_{21}b_1/a_{11} \le 0$ as $b_2/a_{21} \le b_1/a_{11}$. The equilibrium $(b_1/a_{11},0)$ is a **stable node**. In the third case $(b_1/a_{11},0)$ is a **saddle point** as $b_2/a_{21} > b_1/a_{11}$, equivalently $a_{21}/b_2 < a_{11}/b_1$

By symmetry, we have similar conclusions for equilibrium $(0, b_2/a_{22})$: $\lambda_2 = -b_2 < 0$ and $\lambda_1 = b_1 - a_{12}b_2/a_{22} \le 0$ as $b_1/a_{12} \le b_2/a_{22}$, equivalently $a_{22}/b_2 \le a_{12}/b_1$. The equilibrium $(0, b_2/a_{22})$ is a **stable node**. If $b_1/a_{12} > b_2/a_{22}$, equivalently, $a_{22}/b_2 > a_{12}/b_1$ then $(0, b_2/a_{22})$ is a **saddle point**.

The last equilibrium point is the intersection of the lines $b_1 - a_{11}R - a_{12}A = 0$ and $b_2 - a_{21}R - a_{22}A = 0$. If the two lines are not parallel, then the lines have a unique intersection. By comparing the slopes of the two lines, we have the following two cases:

Case I . $M_R > M_A$, i.e., $\dfrac{a_{12}/b_1}{a_{11}/b_1} > \dfrac{a_{22}/b_2}{a_{21}/b_2}$, equivalently, $a_{11}a_{22} < a_{12}a_{21}$, or

Case II . $M_R < M_A$, i.e., $\dfrac{a_{12}/b_1}{a_{11}/b_1} < \dfrac{a_{22}/b_2}{a_{21}/b_2}$, equivalently, $a_{11}a_{22} > a_{12}a_{21}$

However whether or not the location of the intersection is in the first quadrant is determined by more comparisons as shown in Figures (a) through (d) below

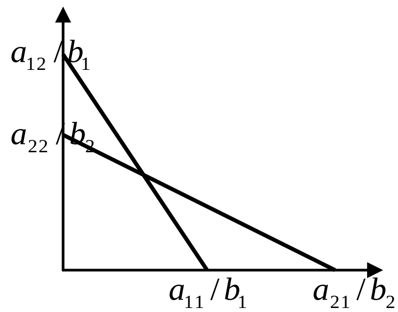


Figure (a)

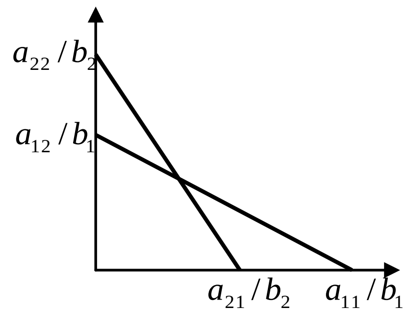


Figure (b)

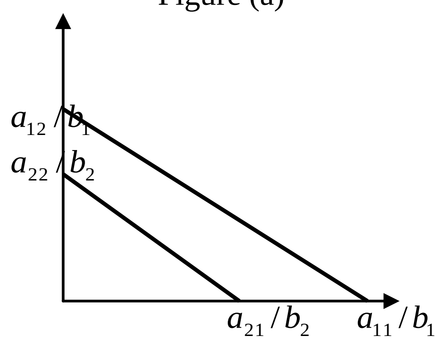


Figure (c)

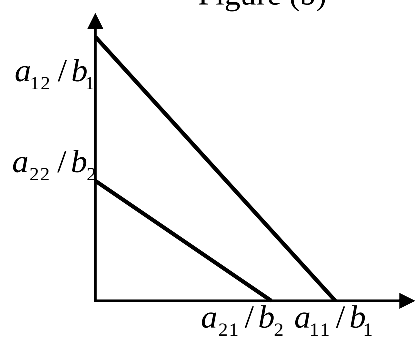


Figure (d)

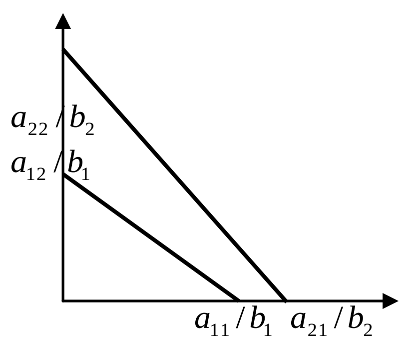


Figure (e)

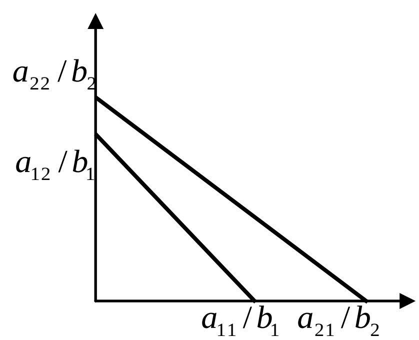


Figure (f)

In Figure (a), it is required that $a_{22}/b_2 < a_{12}/b_1$ and $a_{11}/b_1 < a_{21}/b_2$, equivalently $a_{22}/a_{12} < b_2/b_1$ and $b_2/b_1 < a_{21}/a_{11}$. All three inequalities can be combined into one double inequality $a_{22}/a_{12} < b_2/b_1 < a_{21}/a_{11}$. This inequality is only possible in Case I, $a_{11}a_{22} < a_{12}a_{21}$.

Symmetrically, Figure (b) is held if $a_{22}/a_{12} > b_2/b_1 > a_{21}/a_{11}$, affiliated with Case II, $a_{11}a_{22} > a_{12}a_{21}$.

An analytic solution for the intersection confirms the conclusions for the situation illustrated in Figures (a) and (b). Furthermore, we can us the solution to find out the conditions for the situations when the intersection falls out of the first quadrant. Explicitly, the intersection, in general form, is given by

$$\begin{aligned}\begin{bmatrix} R^* \\ A^* \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 + a_{11}b_2 \end{bmatrix}\end{aligned} \tag{21}$$

For the intersection to be out of the first quadrant, it is necessary that two components are opposite in signs, i.e., $(a_{22}b_1 - a_{12}b_2)(-a_{21}b_1 + a_{11}b_2) < 0$. There are two possibilities in this case: $a_{22}b_1 - a_{12}b_2 > 0 \,\&\, -a_{21}b_1 + a_{11}b_2 < 0$ or $a_{22}b_1 - a_{12}b_2 < 0 \,\&\, -a_{21}b_1 + a_{11}b_2 > 0$

Simple algebraic changes show that $a_{22}b_1 - a_{12}b_2 > 0 \,\&\, -a_{21}b_1 + a_{11}b_2 < 0$ are equivalent to $a_{22}/a_{12} > b_2/b_1 \,\&\, a_{21}/a_{11} > b_2/b_1$. Therefore the intersection is located in the second quadrant in the case $a_{22}b_1 - a_{12}b_2 > 0 \,\&\, -a_{21}b_1 + a_{11}b_2 < 0$ when $a_{11}a_{22} < a_{12}a_{21}$ and in the fourth quadrant when $a_{11}a_{22} > a_{12}a_{21}$.

From the above arguments and using the symmetry of the system, we summarize six possibilities of the location of the non-trivial intersection (R^*, A^*) when the two lines are not parallel.

Table 2. Location criteria of the non-trivial equilibrium

Slope comparison	**Coefficient Response**	**Condition**	**Intersection Location**	**Figure**
$M_A < M_R$	$a_{11}a_{22} < a_{12}a_{21}$	$a_{22}/a_{12} < b_2/b_1 < a_{21}/a_{11}$	**Quad I**	***(a)***
$M_A > M_R$	$a_{11}a_{22} > a_{12}a_{21}$	$a_{22}/a_{12} > b_2/b_1 > a_{21}/a_{11}$	**Quad I**	***(b)***
$M_A < M_R$	$a_{11}a_{22} < a_{12}a_{21}$	$Min\{a_{22}/a_{12}, a_{21}/a_{11}\} > b_2/b_1$	**Quad II**	***(c)***
$M_A < M_R$	$a_{11}a_{22} < a_{12}a_{21}$	$Max\{a_{22}/a_{12}, a_{21}/a_{11}\} < b_2/b_1$	**Quad IV**	***(d)***
$M_A > M_R$	$a_{11}a_{22} > a_{12}a_{21}$	$Min\{a_{22}/a_{12}, a_{21}/a_{11}\} > b_2/b_1$	**Quad IV**	***(e)***
$M_A > M_R$	$a_{11}a_{22} > a_{12}a_{21}$	$Max\{a_{22}/a_{12}, a_{21}/a_{11}\} < b_2/b_1$	**Quad II**	***(f)***

We now deliberate the phase portrait for the first two cases when a positive equilibrium (R^*, A^*) exists.

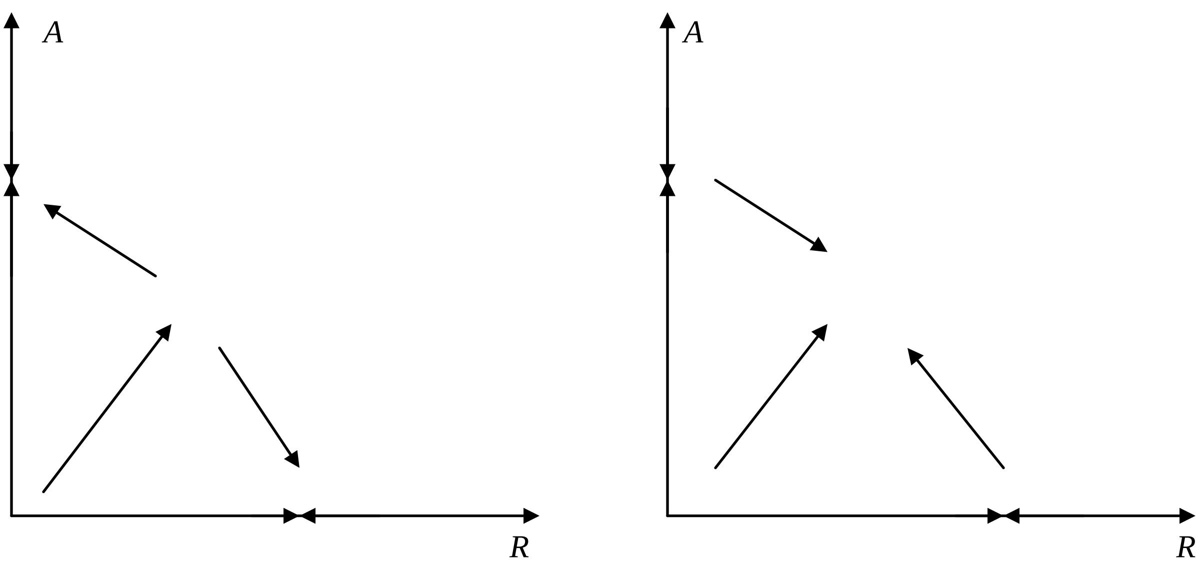
Condition $a_{22}/a_{12} < b_2/b_1 < a_{21}/a_{11}$ implies the existence of a positive equilibrium point (R^*, A^*) and also the coefficient response $a_{11}a_{22} < a_{12}a_{21}$ that suggests strong competitions between the competitors R and A. The positive equilibrium is an unstable node. In other words, either authority or rebellion will eventually win the competition and expand to its maximum. The other side will be defeated.

On the other hand, $a_{22}/a_{12} > b_2/b_1 > a_{21}/a_{11}$ indicates weak competitions so that both can coexist. In the case of weak competition, both sides will coexist in long term run.

We can summarize the classifications of all equilibriums in the following table when the positive equilibrium exists.

Table 3. Behavior near equilibriums

Condition /Figure	**Competing Level**	(0,0)	$(b_1/a_{11},0)$	$(0,b_2/a_{22})$	(R^*, A^*)
$a_{22}/a_{12} < b_2/b_1 < a_{21}/a_{11}$ Figure (a)	Strong	Unstable node	Stable node $a_{21}/b_2 > a_{11}/b_1$	Stable node $a_{22}/b_2 < a_{12}/b_1$	Unstable node
$a_{22}/a_{12} > b_2/b_1 > a_{21}/a_{11}$ Figure (b)	Weak	Unstable node	Saddle point $a_{21}/b_2 < a_{11}/b_1$	Saddle point $a_{22}/b_2 > a_{12}/b_1$	Stable node



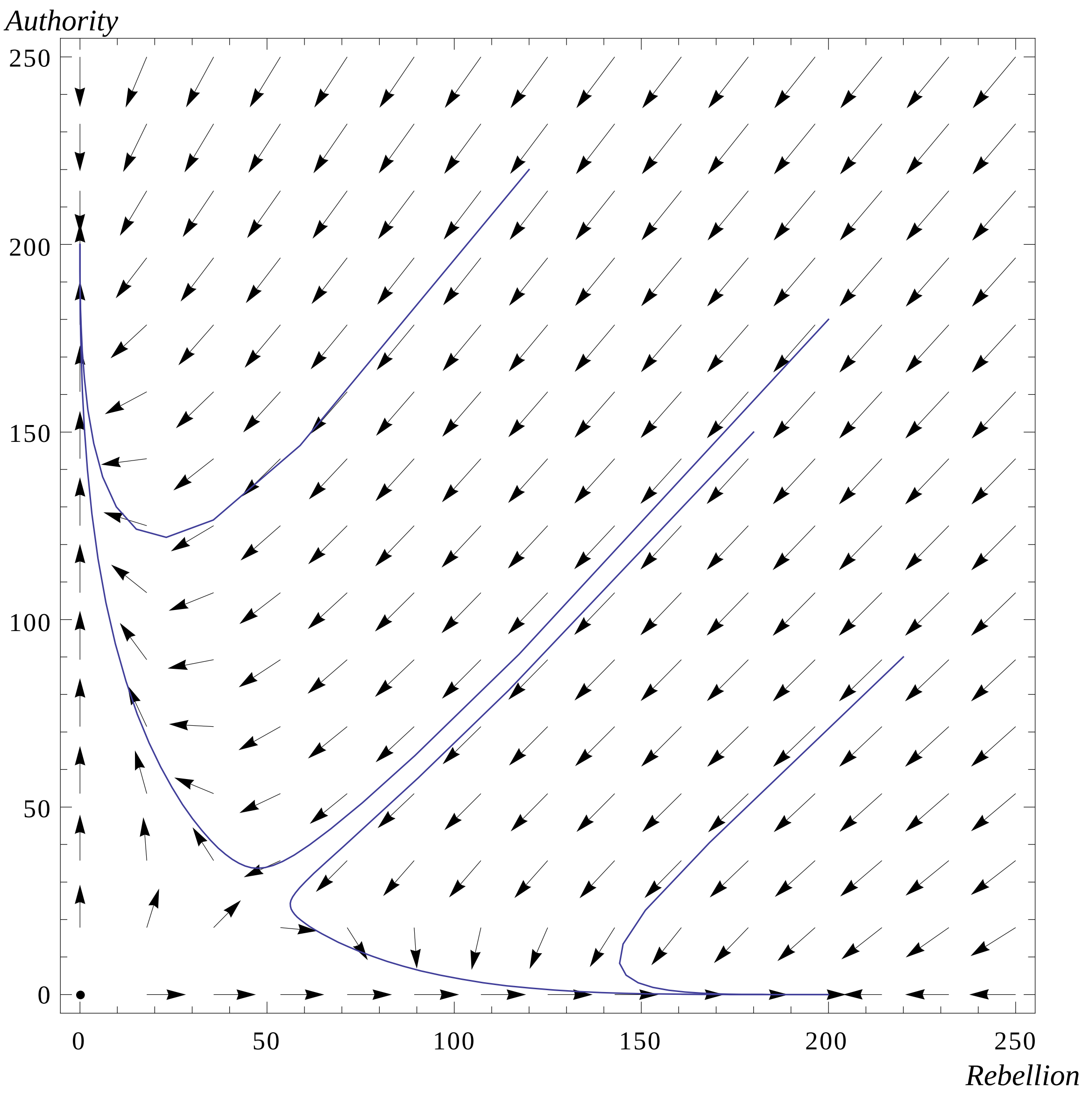
Strong competition **Weak competition**

We now simulate the two cases by applying Mathematica to two selected examples.

Example 4

$$\frac{dR}{dt} = R(2 - 0.01R - 0.06A)$$
$$\frac{dA}{dt} = A(4 - 0.07R - 0.02A) \qquad (22)$$

System (22) is a case of strong competition with equilibriums $(0,0), (200,0), (0,200)$, and $(50,25)$.

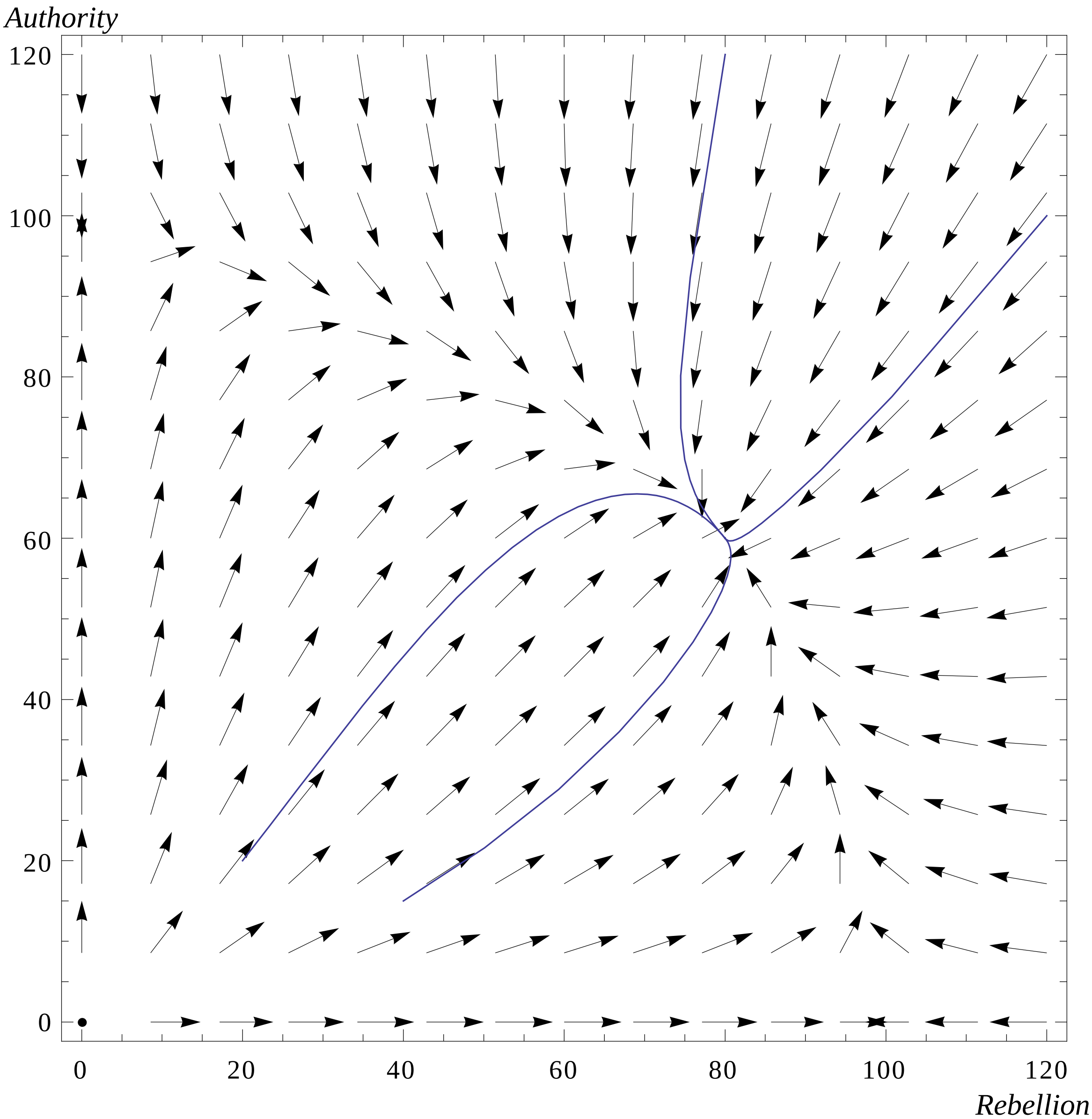


Four trajectories, starting from different positive initial values, turn away from the positive equilibrium and approach either $(200,0)$ or $(0,200)$. This example supports the claim that under strong competition, either rebellion or authority force dies out.

Example 5

$$\frac{dR}{dt} = R(6 - 0.06R - 0.02A)$$
$$\frac{dA}{dt} = A(8 - 0.04R - 0.08A) \qquad (23)$$

System (23) is a case of weak competition with equilibriums $(0,0), (100,0), (0,100)$, and $(80,60)$.



Four trajectories, starting from different positive initial values, approach the positive equilibrium. This example shows that under weak competition, both rebellion and authority forces survive for a relatively long period.

Conclusions and Discussion

In the last two sections, we have proposed the predator-prey type and interference authority-rebellion competition models and conducted preliminary investigations on the extinction and coexistence of authority and rebellion forces.

In the predator-prey type system, under the assumption that the authority has enough backup reserves, we find that if the authority responding ratio *e* is too low, the rebellion finally reaches its maximal allowed capacity from the population. On the other hand, if the neutral population support to the insurgency force is strong enough, coexistence of both forces is observed, no matter what authority military lethality is. The fact reaffirms that winning the neutral population is the key to the counterinsurgency.

In the interference COIN competition models, we observe that under strong competition, one of the two competitors, either authority or rebellion, loses the competition eventually. However coexistence is possible in weak competition level.

The above two kinds of COIN models work fine in simplified situations and can be used for qualitative scenario description. More general models are needed for future case studies.

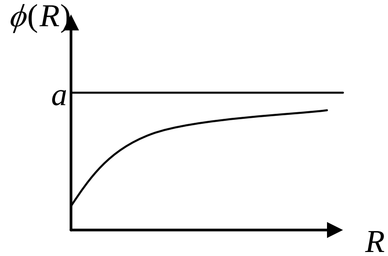
Consider the general rebellion-authority predator-prey type system

$$\begin{aligned} \frac{dR}{dt} &= f(R) - \phi(R,A) \\ \frac{dA}{dt} &= -dA + e\phi(R,A) \end{aligned} \qquad (24)$$

where $f(R)$ is the rate of change in the absence of authority $A(t)$, $\phi(R,A)$ is the rate of predation, d is the mortality rate of the predator, or decreasing rate of authority without $R(t)$, e is the authority's attacking efficiency. We assume that the authority force use is limited only by its rebellion. To make it simpler and easy to handle, we assume that the rate at which authority takes on rebellion depends only on the rebellion density, and independent of the density of authority. Then System (24) becomes

$$\begin{aligned} \frac{dR}{dt} &= f(R) - A\phi(R) \\ \frac{dA}{dt} &= -dA + eA\phi(R) \end{aligned} \qquad (25)$$

where $\phi(R)$ is called the functional response of predators. It is reasonable to assume that the response is zero with no appearance of rebellion force and growing steadily to its maximal lethality level as the rebellion force increases. As an attempt, we adopt so called Holling Type II response (also known as Michaelis-Menton response), a proved popular response in ecology, and assume that $\phi(R)$ satisfies the following conditions: $\phi(0) = 0, \phi'(R) > 0, \phi''(R) < 0$. An example of the Holling Type II response function is $\phi(R) = \dfrac{aR}{1+R}$



Holling Type II response function can also be applied to interference competition models. We noticed that in the interference competition model, the interaction among the competitors and the neutral population is not involved. Gen. Rene Emilio Ponce, Defense Minister of El Salvador during the 1980's, was quoted that 90% of the country's counter insurgency success is political, social, economic and ideological and only 10% military (B. Hoffman (2004)). Similarly Galula (1964) proposed the 80/20 Rule (D. Kilcullen (2007)): Essential though it is, the military action is secondary to the political one, its primary purpose being to afford the political power enough freedom to work safely with the population. A revolutionary war is 20% military action and 80% political is a formula that reflects the truth.

According to the RAND research data of six selected COIN case studies (A. Rabasa et al (2007), Table S.1), high ratio of population supporting COIN (therefore low ratio supporting insurgency), not the military lethality, is the most important key factor to COIN wins.

Taking into consideration winning the neutral population in a counterinsurgency operation, we should consider adopting an additional equation to address the rate of change of the neutral population in a study of modeling interference COIN competitions. For example, we can adopt the following system of equations to simulate the interactions among the neutral population (P), rebellion (R) and authority (A)

$$\begin{aligned} \frac{dP}{dt} &= rP\left(1-\frac{P}{K}\right) - R\phi_1(P) - A\phi_2(P) \\ \frac{dR}{dt} &= R[b_1 - a_{11}R - a_{21}A + \phi_1(P)] \\ \frac{dA}{dt} &= A[b_2 - a_{21}R - a_{22}A + \phi_2(P)] \end{aligned} \qquad (26)$$

where *P*, *R* and *A* denote the densities of the neutral population, rebellion force and authority force, respectively; *r* is the maximal intrinsic growth rate of the neutral population; $\phi_1(P)$ and $\phi_2(P)$ are response functions from rebellion and authority forces, respectively; *K* is the maximal capacity of the neutral population. We can also adopt Holling II responses for System (26)

$$
\begin{aligned}
\frac{dP}{dt} &= rP\left(1-\frac{P}{K}\right)-\frac{aRP}{1+bP}-\frac{cAP}{1+dP} \\
\frac{dR}{dt} &= R\left[b_1 - a_{11}R - a_{21}A + \frac{aP}{1+bP}\right] \\
\frac{dA}{dt} &= A\left[b_2 - a_{21}R - a_{22}A + \frac{cP}{1+dP}\right]
\end{aligned} \tag{27}
$$

For future case studies of COIN operations using interference competition models, we must be cautious on the following limitations and assumptions:

- The whole system must be closed with no migration.
- Majority of a population are neutral.
- Reliable COIN data are hard to obtain.

The last but important is that mathematical models are not reality due to oversimplifications and therefore can only serve as an admittedly crude framework for understanding fundamental components of COIN warfare (Drapeau et al 2008).

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